Generalized Barycentric Coordinates via Harmonic Weights

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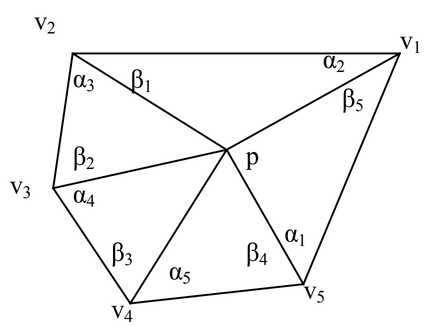
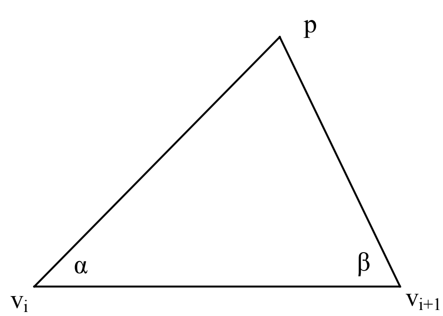
From time to time the problem arises of finding "barycentric" coordinates for points relative to an n-gon. When n>3, there is no unique set of barycentric coordinates - the problem is under-determined. However, a set of weights used in discrete harmonic mapping theory turns out to be useful.



For a point p and a planar n-gon whose vertices are $(v_1,...,v_n)$, we claim that

$$p = \frac{1}{\sum_i (\cot\alpha_i + \cot\beta_i)} \sum_i (\cot\alpha_i + \cot\beta_i) v_i \, . \qquad (1)$$

To prove this we will consider the planar triangle (p, v_i, v_{i+1}):



Let $\alpha = \angle p v_i v_{i+1}$ and $\beta = \angle v_i v_{i+1} p$, and let ρ be a clockwise rotation by 90 degrees.

Lemma: $(\cot\alpha)(v_{i+1} - p) + (\cot\beta)(v_i - p) = \rho(v_{i+1} - v_i)$.

The proof can be found in Figure 2 of [PP].

The main result follows from this, because (all subscripts are modulo n)

$$\begin{aligned}
0 &= \rho(0) \\
&= \rho\Big(\sum_i (v_{i+1} - v_i)\Big) \\
&= \sum_i \rho(v_{i+1} - v_i) \\
&= \sum_i \cot\alpha_{i+1}(v_{i+1} - p) + \cot\beta_i (v_i - p) \\
&= \sum_i (\cot\alpha_i + \cot\beta_i)(v_i - p) \\
&= \sum_i (\cot\alpha_i + \cot\beta_i) v_i - \sum_i (\cot\alpha_i + \cot\beta_i) p
\end{aligned}$$

from which (1) follows.

An equivalent expression of these weights, in terms of areas and edge-lengths, can be found in [Eck+, Eqn 1].

These weights may be used as a way of mapping the interior points between two n-gons. They were originally used to calculate discrete harmonic mappings on meshes, and have also been used for mesh smoothing and calculating "discrete normals"[DMSB]. Indeed, for p not in the plane of $(v_1,...,v_n)$, the above formula can be thought of as the projection of p onto $(v_1,...,v_n)$.

References

[DMSB] Mathieu Desbrun, Mark Meyer, Peter Schröder and Alan H. Barr. Implicit Fairing of Irregular Meshes using Diffusion and Curvature Flow. *SIGGRAPH 99 Conference Proceedings*, pages 317-324. Available at http://www.cs.caltech.edu/~mmeyer/Research/Projects/FairMesh/implfair.pdf.

[Eck+] M. Eck, T. DeRose, T. D., H. Hoppe, M. Lounsbery, and W. Stuetzle. Multiresolution analysis of arbitrary meshes. In Robert Cook, editor, SIGGRAPH 95 Conference Proceedings, Annual Conference Series, pages 173--182. ACM SIGGRAPH, Addison Wesley, August 1995. held in Los Angeles, California, 06-11 August 1995. Available at http://www.stat.washington.edu/wxs/Siggraph-95/siggraph95.pdf.

[PP] Ulrich Pinkall and Konrad Polthier, Computing Discrete Minimal Surfaces and Their Conjugates. Experimental Mathematics, **2**, 1993. Available at http://www-sfb288.math.tu-berlin.de/~konrad/articles/diri/diri_jem.pdf.